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T-dependent Dyson–Schwinger equation in IR regime of QCD: the critical point

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Abstract. The quark mass function $\Sigma(p)$ in QCD is revisited, using a gluon propagator in the form $1/(k^2 + m_g^2)$ plus $2\mu^2/(k^2 + m_g^2)^2$, where the second (IR) term gives linear confinement for $m_g = 0$ in the instantaneous limit, μ being another scale. To find $\Sigma(p)$ we propose a new (differential) form of the Dyson–Schwinger equation (DSE) for $\Sigma(p)$, based on an infinitesimal *subtractive* renormalization via a differential operator which *lowers* the degree of divergence in integration on the RHS, by *two* units. This warrants $\Sigma(p-k) \approx \Sigma(p)$ in the integrand since its k-dependence is no longer sensitive to the principal term $(p-k)^2$ in the quark propagator. The simplified DSE (which incorporates the Ward–Takahashi (WT) identity in the Landau gauge) is satisfied for large p^2 by $\Sigma(p) = \Sigma(0)/(1+\beta p^2)$, except for Log factors. The limit $p^2 = 0$ determines Σ_0. A third limit, $p^2 = -m_0^2$, defines the *dynamical* mass m_0 via $\Sigma(im_0) = +m_0$. After two checks ($f_\pi = 93 \pm 1\,\text{MeV}$ and $\langle q\bar{q}\rangle = (280 \pm 5\,\text{MeV})^3$), for $1.5 < \beta < 2$ with $\Sigma_0 = 300\,\text{MeV}$, the T-dependent DSE is used in the real time formalism to determine the "critical" index $\gamma = 1/3$ analytically, with the IR term partly serving as the H-field. We find $T_c = 180 \pm 20\,\text{MeV}$ and check the vanishing of f_π and $\langle q\bar{q}\rangle$ at T_c.

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1 Introduction

QCD, as the queen of the strong interaction theory, lies at the root of a whole complex of strong interaction phenomena, ranging from particle physics to cosmology. Its principal tool is the quark mass function, termed $\Sigma(p)$ in the following, being a central ingredient for the evaluation of a string of QCD parameters whose primary examples are the pion decay constant and the quark condensate. The thermal behavior of the latter in turn has acquired considerable cosmological relevance in recent years in the context of global experimentation on heavy ion collisions as a means of accessing the quark–gluon plasma (QGP) phase [1–4]. It is therefore essential to have at hand a reliable $\Sigma(p)$ function in a non-perturbative form as a first step towards the evaluation of these basic QCD parameters. In this respect, QCD sum rules (SR), attuned to finite temperatures [2] have been a leading candidate for such studies for a long time, using the FESR duality principle [5], as well as a variational approach via the minimum of the effective action up to two loops (see Barducci et al. in the first reference of [1]) to determine the mass function. An alternative approach has been the method of chiral perturbation theory [6] with the *pion* as the basic unit in preference to quarks. Now a standard approach to QCD is via the RG equation for the β function in the lowest order of g which yields

$$\alpha_s(Q^2) = 2\pi/[9\ln(Q/\Lambda_Q)]$$

with three flavors, Λ_Q being the QCD scale parameter [7]. Unfortunately the higher order terms in g are not particularly amenable to the simulation of non-perturbative effects. On the other hand, the Dyson–Schwinger equation (DSE), which may be regarded as the differential form of the minimum principle of the effective action [8], offers us a more promising tool which has often been used with the standard one-gluon exchange (OGE) in the rainbow approximation [9] but can be improved to incorporate gauge invariance so as to satisfy the WT identity in the "dynamical perturbation theory" (which ignores criss-cross gluon lines in the skeleton diagrams) with little extra effort, as first shown by Pagels–Stokar [10]. In this paper, we shall use the same approach, but explicitly add an extra, non-perturbative, term to the one-gluon-exchange (OGE) propagator for a quicker simulation of the infrared (IR) regime, so that both together act as the "kernel" of the Dyson–Schwinger equation (DSE) [11]. Thus the total gluon propagator is given by

$$G(k) = \frac{1}{k^2 + m_g^2} + \frac{2\mu^2}{(k^2 + m_g^2)^2}\,, \tag{1.1}$$

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where μ is a scale parameter corresponding to the (hadronic) GeV regime (whose value will be left undetermined until later), and m_g is a (small) gluon mass with a non-zero value, which can be motivated from several angles, a notable one being the "Schwinger mechanism" [12] as explained in the Jackiw–Johnson paper [13]. A second motivation was highlighted by Cornwall et al. [14], in the context of their approach to a more compact realization of gauge invariance via the so-called "pinch mechanism" [14]. Yet a third motivation, which is especially relevant in the present context of a temperature dependent DSE, comes from according to it a "Debye mass" status, running with the temperature [15]. A non-perturbative gluon propagator (with harmonic confinement) was employed in [16,17] as a kernel of a BSE for the gg wave function for the calculation of glueball spectra, on lines similar to $q\bar{q}$ spectroscopy [18]. Alternative BSE treatments for glueballs also exist in the literature [19]. In this paper, the IR part of (1.1) has a dual role:

(1) to serve as a more efficient simulation of the non-perturbative effects on the mass function $\Sigma(p)$, and

(2) a partial simulation of the external magnetic field effect, as an alternative to small non-zero masses of "current" quarks [15,20]. With a non-perturbative solution of the DSE, we are primarily concerned with chiral symmetry restoration at a critical temperature T_c. To that end we shall be interested in the T behavior near the critical point T_c, rather than as an expansion in powers of T^2 near $T = 0$ [6]. Note however that linear confinement ($\sim r$) corresponds to $m_g = 0$, via the second term in (1.1), in the (3D) instantaneous limit $t = 0$, so that deconfinement competes with chiral symmetry restoration with a propagator like (1.1). We shall not pursue this aspect further, although we note that deconfinement has been claimed to occur at a lower temperature [21] than chiral symmetry restoration.

1.1 Object and scope of this paper

The central object of this paper is a determination of the mass function $\Sigma(p)$ non-perturbatively in the intermediate momentum regime with the help of the gluon propagator (1.1) that covers the IR regime. This is sought to be achieved via a (new) differential formulation of the DSE based on a subtractive form of renormalization that is particularly convenient for a DSE type equation. A second object is to apply the $\Sigma(p)$ so determined to two basic quantities, $\langle q\bar{q}\rangle$ and f_π, and express them in an *analytic* form, so that their T-dependent generalizations may be achieved analytically too. A third object is to generalize the DSE to a T-dependent form, so as to obtain an equation for the T-dependent mass function m_t, with a focus on its critical index γ associated with the critical temperature T_c, so as to gauge the role of the IR term vis-à-vis small current masses to simulate the H-field effect [15,20]. Further, while in the conventional methods [15,20] the various thermodynamic quantities are derived from a central quantity like the free energy [15], or equivalently the effective potential [20], and taking appropriate derivatives, the plan adopted here is to focus on the DSE itself as the principal form of dynamics, with m_t as a natural order parameter. Due to the unconventional nature of this approach, this part of the exercise is still preliminary, with only one critical index identifiable with m_t determined so far, while the other indices [20] are left for later studies, within the DSE framework.

In Sect. 2, we formulate the DSE for $\Sigma(p)$ in an (infinitesimal) form of (subtractive) renormalization which yields a non-linear second-order differential equation for this quantity. The *dynamical* mass m_0 is defined as the pole of $S_F(p)$ at $i\gamma.p = -m_0$ and hence corresponds to the solution of the equation for $\Sigma(im_0) = m_0$. Although in principle a mass renormalization factor Z_m comes according to the rules of [7], the condition $\Sigma(im_0) = m_0$ ensures that this factor is effectively unity, provided the dynamical mass is employed for the propagator at its pole. As for the quantity $\Sigma(0)$, we shall designate it as the *constituent* mass. For the solution of the resulting DSE, three crucial check-points are $p^2 = \infty$; $p^2 = 0$; $p^2 = -m_0^2$ which control the structure of $\Sigma(p)$. The simplest ansatz consistent with a p^{-2}-like behavior in the $p^2 = \infty$ limit, as demanded by QCD, is $\Sigma_0/(1+\beta p^2)$ [1], the only precaution needed for a consistent solution being a constant α_s with its argument *fixed* in advance at a certain specified value. This form has good analytical properties for large space-like momenta, but it implies that the dynamical mass $\Sigma(im_0)$ exceeds the constituent mass $\Sigma(0)$.

For a basic test of this structure, we choose in Sect. 3 two key items:

(i) $q\bar{q}$ and

(ii) f_π^2, whose derivations are sketched in Appendices A and B respectively in an *analytical* form. The results agree with experiment to within $\sim 5\%$, for $\Sigma_0 = 300\,\mathrm{MeV}$, $m_g \approx \Lambda_Q = 150\,\mathrm{MeV}$, and the hadronic scale parameter β in the range $1.5 < \beta < 2.0$.

Section 4 outlines the formulation of the temperature dependent DSE (T-DSE for short) within the real time formalism [22], instead of the imaginary time formalism à la Matsubara [23]. The order parameter in this regard may be chosen in one or more ways, a convenient choice being Σ_0 which now "runs" with the temperature and is renamed m_t. Other analogous quantities which are expected to "run" with the temperature are the gluon mass renamed as m_{gt}, and perhaps also the IR parameter μ whose connection with m_t and m_{gt} is brought out in Sect. 4. It is found that the constituent and gluon masses have the *same* "critical index" $\gamma = 1/3$ (in accordance with the concept of "universality" of the critical indices), while the critical temperature works out as $T_c \approx 180 \pm 20\,\mathrm{MeV}$. Section 5 concludes with a discussion including a comparison with contemporary approaches.

2 Dyson–Schwinger equation in differential form

We start by writing the DSE in the Landau gauge which ensures that the A parameter does not suffer renormalization [24]. This is an additional precaution over and above the Pagels–Stokar DPT approach [10] to satisfy the WT identity. The starting DSE in the Landau gauge for the function $\Sigma(p)$, after tracing out the Dirac matrices takes the form

$$\Sigma(p) = \mathrm{i}g_s^2 \frac{F_1.F_2}{(2\pi)^4} \int \mathrm{d}^4k \frac{1}{[\Sigma^2(p-k)+(p-k)^2]} \tag{2.1}$$
$$\times\left[\Sigma(p-k)\delta_{\mu\nu} + (\Sigma(p)-\Sigma(p-k))\frac{(p-k)_\mu(2p-k)_\nu}{k^2-2p.k}\right]$$
$$\times(\delta_{\mu\nu} - k_\mu k_\nu/k^2)\left[\frac{1}{k^2+m_g^2} + \frac{2\mu^2}{(k^2+m_g^2)^2}\right].$$

The first term on the RHS corresponds to the rainbow approximation [9], while the second term gives the simplest realization of a gauge invariant structure by satisfying the WT identity à la Pagels–Stokar [10]. An analogous but slightly more involved ansatz due to Ball–Chiu [25] also can be seen to conform to the Landau form [10], through a visual inspection of both. To see this more explicitly, we list both forms for the relevant vertex functions, first [25] (as given in [24]) followed by [10]

$$\Gamma_\nu(p',p) = -\mathrm{i}\gamma_\nu[A+A']/2$$
$$+\frac{A'-A}{2(p^2-p'^2)}[-\mathrm{i}\gamma.(p+p')(p_\nu+p'_\nu)]$$
$$+\frac{B-B'}{p^2-p'^2}(p_\nu+p'_\nu), \tag{2.2}$$
$$\Gamma_\nu(p',p) = -\mathrm{i}\gamma_\nu + \frac{\Sigma(p)-\Sigma(p')}{p^2-p'^2}(p_\nu+p'_\nu),$$

where the momentum dependence (p,p') of the Ball–Chiu functions A,B is indicated by the unprimed and primed notation respectively and the mass function $\Sigma(p)=B/A$, while the Landau gauge corresponds to $A=1$. The Ball–Chiu form [25] is seen to be compatible with Pagels–Stokar [10] (which is already in the Landau gauge, $A=1$). So, without further ado, we shall use only [10] for simplicity.

We now adopt a *subtractive* form of renormalization by writing a similar equation for, say, p', and *subtracting* one from the other. If p' is infinitely close to p, this results in a differential form. Thus we subject both sides of (2.1) to the differential operator $p.\partial$, *not* the scalar form $p^2\partial_{p^2}$, since the former is more naturally attuned to handling *two* vectors p,k that occur on the RHS. The main advantage of this crucial step is to reduce the degree of divergence of the integral with respect to k by *two* units, which in turn allows for further simplifications on $\Sigma(p-k)$ on the RHS, since it falls off rapidly with k^2. In particular, we are allowed to make the following simplification as a result of this crucial step of reducing the divergence via differentiation:

$$\frac{\Sigma(p)-\Sigma(p-k)}{k^2-2p.k} \approx -\partial_{p^2}\Sigma(p).$$

A second simplification arises from a contraction of the factors $(p-k)_\mu(2p-k)_\nu$ and $(\delta_{mu\nu} - k_\mu k_\nu/k^2)$ which is almost *independent* of k_μ and gives on angular integration [26]

$$2[p^2-(p.k)^2/k^2] \approx 2(1-n^{-1})p^2; \quad n=4.$$

Further, against the background of the differential operator $p.\partial_p$ on both sides of (2.1), we can replace the mass function $\Sigma^2(p-k)$ inside the fermion propagator on the RHS due to an improved k-convergence, by simply replacing $\Sigma^2(p-k)$ with $\Sigma^2(p)$, since this quantity already falls off with momentum (see also [10]). The resulting equation (2.1) now takes the form

$$2\Sigma'(p) = \frac{4g_s^2}{\mathrm{i}(2\pi)^4}\int \mathrm{d}^4k\left[\frac{2\Sigma'(p)-\Sigma''(p)}{D(p-k)}\right.$$
$$\left.-\frac{\Sigma(p)-\Sigma'(p)/2}{D^2(p-k)}(4\Sigma(p)\Sigma'(p)+2p^2-2p.k)\right]$$
$$\times\left[\frac{1}{k^2+m_g^2}+\frac{2\mu^2}{(k^2+m_g^2)^2}\right], \tag{2.3}$$

where we have taken $F_1.F_2=-4/3$, and we defined derivatives and propagators as

$$\Sigma'(p) = (1/2)p.\partial_p\Sigma(p) = p^2\partial_{p^2}\Sigma(p);$$
$$D(p-k) = \Sigma^2(p)+(p-k)^2. \tag{2.4}$$

Note that decoupling of $\Sigma(p)$ from k_μ now facilitates the k-integration, thus converting the DSE into a differential equation, while the form of $\Sigma(p)$ is as yet undetermined. (This structure is different from a more conventional one for a differential form of the DSE, by making the $D(p-k)$ separable in terms of $p_>$ and $p_<$, etc. [7,14].)

The next task is to integrate with respect to d^4k, which for the OGE term is still logarithmically divergent and hence requires “dimensional regularization ” (DR) à la ‘t Hooft–Veltman [27], while the IR term gives a convergent integral. We hasten to add that this divergence (despite the Landau gauge) may well be an artefact of the approximation $\Sigma(p-k)\approx\Sigma(p)$ in the numerator on the RHS, but since the divergence is only logarithmic, it is not sensitive to DR [27], and in any case it is a small price to pay for the huge advantage accruing from this (new) differentiation method for renormalization. Another approximation concerns the factor g_s^2 on the RHS of (2.1) and (2.2) which, strictly speaking, is a function of the momenta p,k, but at this stage we must “freeze” the value of α_s at a *fixed* value (to be specified below) so as to get a self-consistent asymptotic solution in the $p^2=\infty$ limit. [More general solutions with the differential form (2.2) and variable α_s have not been attempted here.]

2.1 Dimensional regularization for integrals

Denote the two integrals of (2.2) containing the OGE term only by I and II respectively, of which only I is divergent (see above), but II is convergent by itself. Thus write for I in the Euclidean notation for dimension n, using the DR method [11,27],

$$I = 4g_s^2(2\Sigma'(p)-\Sigma''(p))\zeta^\epsilon \tag{2.5}$$
$$\times\int\frac{\mathrm{d}^nk}{(2\pi)^nD(p-k)(k^2+m_g^2)}$$

$$= 4g_s^2(2\Sigma'(p) - \Sigma''(p))\zeta^\epsilon \times \int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}k^2 k^{n-2} \frac{\pi^{n/2}}{\Gamma(n/2)(2\pi)^n(\Lambda_u + k^2)^2} ,$$

where we have introduced the Feynman variable $0 \le u \le 1$, ζ is a UV dimensional constant, $\epsilon = 4 - n$, and

$$\Lambda_u = u\Sigma^2(p) + p^2u(1-u) + m_g^2(1-u) .$$

The integration over k^2 is now straightforward, while that over u is simplified by dropping the m_g^2 term since there is no infrared divergence. The result of all these steps after subtracting the UV divergence [27] is (with $g_s^2 = 4\pi\alpha_s$):

$$I = (\alpha_s/\pi)(2\Sigma'(p) - \Sigma''(p))[\ln 4\pi - \gamma + 1 + \ln(\zeta^2/A_p)] , \tag{2.6}$$

where

$$A_p \equiv \Sigma^2(p) + p^2/2 . \tag{2.7}$$

The other integral, II, which is UV convergent does not need DR [27] and gives

$$II = -(\alpha_s/\pi)(\Sigma(p) - \Sigma'(p)/2)\frac{(4\Sigma(p)\Sigma'(p) + p^2)}{A_p} . \tag{2.8}$$

Thus the resulting DSE may be expressed compactly from (2.3) as

$$2\Sigma'(p) = I + II + I' + II' , \tag{2.9}$$

where we have taken I and II from the OGE contributions (2.7) and (2.8) respectively as well as added two similar (infrared) terms I' and II' arising from the second (IR) part of the gluon propagator (1.1). In the same normalization as above the last two work out as

$$I' = \frac{2\mu^2\alpha_s}{\pi A_p}(2\Sigma'(p) - \Sigma''(p))[\ln(A_p/m_g^2) - 1] ,$$

$$II' = -\frac{2\mu^2\alpha_s}{\pi A_p^2}(\Sigma(p) - \Sigma'(p)/2)(4\Sigma(p)\Sigma'(p) + p^2) \times[\ln(A_p/m_g^2) - 2] , \tag{2.10}$$

where we have made use of the smallness of m_g which is simplifying for some of the integrals over u. Note that the last two terms are at least of two lower orders in p than their OGE counterparts, so that they will not contribute to the $p^2 = \infty$ limit of the differential equation (2.9).

2.2 Large and small p^2 limits of DSE for $T = 0$

To solve (2.9), we try the ansatz [1,20]

$$\Sigma(p) = \Sigma_0/[1 + \beta p^2] , \tag{2.11}$$

whose asymptotic form is compatible with perturbative QCD expectations for massless quarks in the chiral limit [10]. And we take the *fixed* value of p^2 in the argument of α_s at $p^2 = \zeta^2$, where ζ is the UV parameter corresponding to the upper limit of p^2 allowed in the solution of the DSE. [Other options exist but are not particularly convenient.]

2.2.1 Large p^2 limit

Remembering the definition (2.4) for Σ', etc., we have in the large p^2 limit for the function (2.11)

$$\Sigma(p) \approx -\Sigma'(p) \approx +\Sigma''(p) .$$

Now remembering the upper limit of p^2 being constrained by the UV parameter ζ^2, substitution from (2.9) yields the result

$$-2 = \alpha_s/\pi[-3(\ln(4\pi) - \gamma + 1 + \ln 2) - 3] ;$$
$$\pi/\alpha_s = \frac{9}{2}\ln(\zeta/\Lambda_Q) , \tag{2.12}$$

$\Lambda_Q = 150\,\mathrm{MeV}$ being the usual QCD scale parameter. Thus (2.12) determines the value of the maximum momentum ζ within this approach, and shows that our formalism does not permit p^2 to exceed ζ^2. Unfortunately (2.12), which corresponds to the check-point $p^2 = \infty$, restricts ζ to a rather low value:

$$\zeta/\Lambda_Q = 1.5490 ; \quad \zeta = 0.706\,\mathrm{GeV}\ \text{only} , \tag{2.13}$$

where the MS scheme (not $\bar{\mathrm{M}}\mathrm{S}$ [28]) has been employed.

2.2.2 Small p^2 limit

Next we consider the small p^2 limit of (2.9) where for the (fixed) argument of α_s we continue (for mathematical consistency) to maintain the same value of α_s corresponding to $p^2 = \zeta^2$, leading after straightforward simplifications to

$$C_0 + \ln x_1/x_0 - 3 + 1/x_0 + [I' + II'] = 9\ln(\zeta/\Lambda_Q) ;$$
$$C_0 \equiv \ln 4\pi - \gamma = 1.9538 , \tag{2.14}$$

where the dimensionless quantities are defined as

$$x_1 \equiv \zeta^2\beta ; \quad x_0 \equiv \Sigma_0^2\beta . \tag{2.15}$$

Note that (2.14) has a big term on the RHS, viz., 9×1.5490, needing a corresponding augmenting of the LHS, which can come only from the IR terms from (2.9), symbolically denoted by $[I' + II']$ in (2.14), that include the (as yet free) parameter $\lambda = 2\beta\mu^2$. [Of course these IR terms do *not* contribute to (2.12).]

2.3 Dynamical mass and mass renormalization

The third point, $p^2 = -m_0^2$, which defines the dynamical mass, corresponds to the "pole" of the propagator $S_F(p)$, so that

$$\Sigma^2(\mathrm{i}m_0) = +m_0^2 > \Sigma^2(0) . \tag{2.16}$$

It may be recalled that a distinction between the dynamical and constituent masses already exists in the literature. Thus in the notation of Domb [29] (pp. 322–324), p and p_0 correspond to m_0 and Σ_0 respectively.

Substituting from (2.11) gives a cubic equation in m_0^2:

$$\Sigma_0^2 = m_0^2(1-\beta m_0^2)^2\,, \tag{2.17}$$

which implies that $\Sigma_0 < m_0$. Using the dimensionless variables $x_0 = \beta\Sigma_0^2$ and $y_0 = \beta m_0^2$, this reduces to the cubic $y_0(1-y_0)^2 = x_0$ which has at least one real solution for y_0 in terms of x_0:

$$\beta m_0^2 \equiv y_0 \tag{2.18}$$

$$= 2/3 + \sum_{\pm}\left[x_0/2 - 1/27 \pm \sqrt{x_0^2/4 - x_0/27}\right]^{1/3},$$

whose nature can be seen as follows. For small x_0, y_0 is also small (seen directly from the cubic form), but as x_0 increases, y_0 increases more rapidly, until x_0 reaches a critical value $x_{\rm c} = 4/27$ (as seen from (2.18)). Beyond this point y_0 increases more slowly with x_0. The corresponding critical value of β is

$$\beta_{\rm c} = 4/(27\Sigma_0^2) \approx 1.646\,\mathrm{GeV}^{-2},\ \text{for } \Sigma_0 = 300\,\mathrm{MeV}\,. \tag{2.19}$$

We shall keep Σ_0 fixed at 300 MeV but vary β in the typical hadronic range $1.0 < \beta < 2.0$ for applications to key QCD parameters like $\langle q\bar{q}\rangle$ and f_π^2. Now the propagator may be written as

$$S_{\rm FR}(p) = Z_m \frac{\Sigma(p) - \mathrm{i}\gamma.p}{\Sigma^2(p) + p^2}\,, \tag{2.20}$$

making use of (2.17), and formally introducing a "mass renormalization" factor Z_m to be determined. However, using the condition (2.16) in the numerator and denominator of (2.19) shows immediately that near the pole the RHS already has the correct structure, $(m_0 - \mathrm{i}\gamma.p)/[m_0^2 + p^2]$, which suggests that $Z_m = 1$! On the other hand an alternative way to extract the factor $(p^2 + m_0^2)$ from (2.19) suggests a non-zero value of Z_m. This is seen from rewriting the RHS of (2.20) as

$$Z_m(\Sigma(p) - \mathrm{i}\gamma.p)/[\Sigma^2(p) - \Sigma^2(\mathrm{i}m) + m_0^2 + p^2]$$

and taking the limit $p^2 \to -m_0^2$ after extracting the factor $(m_0^2 + p^2)$ from the denominator. Z_m is now determined by the condition that at the *pole* this quantity reduces exactly to $1/(m_0 + \mathrm{i}\gamma.p)$. This gives

$$Z_m = (1 - 3m_0^2\beta)/(1 - m_0^2\beta)\,. \tag{2.21}$$

In view of this ambiguity in the working definition of Z_m, it is not clear if this (finite) Z_m is significant beyond unity. However within this *subtractive* renormalization approach to the DSE, the divergences are already toned down to the logarithmic level, so that renormalization is probably less significant than for the usual (unsubtracted) DSE form. For the rest therefore we shall set $Z_m = 1$ in what follows.

2.4 Solution of (2.9), including IR terms

Taking account of the IR terms in (2.9) , the full equation (2.14) reads

$$0 = Ax_0^2 - x_0(B\lambda + 1) + C\lambda\,, \tag{2.22}$$

$$A = -\ln 4\pi + \gamma + 3 - 2\ln(\zeta/\Sigma_0) + 9\ln(\zeta/\Lambda_Q)\,,$$

$$B = 7 - 6\ln(\Sigma_0/m_g)\,; \quad C = 2 - 2\ln(\Sigma_0/m_g)\,,$$

where $x_0 = \beta\Sigma_0^2$, and $\lambda = 2\mu^2\beta$. A practical way is to solve (2.22) for λ with $\Sigma_0 = 300\,\mathrm{MeV}$ and $m_g = 150\,\mathrm{MeV}$. This gives λ for typical values of the "range parameter" β. Note that the connection between x_0 and y_0 is already determined by (2.17) and (2.18). Now with $\Sigma = 300\,\mathrm{MeV}$, the typical value $\beta = 1\,\mathrm{GeV}^{-2}$ ($x_0 = 0.09$ and $y_0 = 0.115$) yields $\lambda = -0.0640$. The latter is an index of the strength of a (small) IR term needed to provide a self-consistent solution of the DSE in the low momentum regime to match its solution for 'large" momenta. We shall come back to these quantities in Sect. 4 for the T-dependent DSE.

At this stage it may be asked what happened to the third check-point $p^2 = -m_0^2$ for the DSE, analogously to the points $p^2 = \infty$ and $p^2 = 0$ considered in the foregoing. As a matter of fact, this condition has already been subsumed in the determination of the relation between the constituent and dynamical masses in (2.17) and (2.18) within the specific structure (2.11), so no new results can be expected from the DSE for $p^2 = -m_0^2$. The check-point $p^2 = -m_0^2$ will however come into play again in Sect. 4, but in a T-dependent form of the DSE. But before implementing the T-dependent DSE programme, it is first necessary to carry out *two* vital tests of this $T = 0$ formalism, viz., its performance on the two crucial quantities $\langle q\bar{q}\rangle$ and f_π^2 [1,5,6], which we consider next.

3 Tests of mass function: $\langle q\bar{q}\rangle$ and f_π^2

The quark condensate and the pion decay constant are regarded as fairly sensitive tests of the mass function $\Sigma(p)$ determined as a solution of the DSE, expressed in the differential form (2.9). To that end we first collect their formal definitions as follows. The condensate after tracing out the Dirac matrices is

$$\langle q\bar{q}\rangle_0 = \frac{4N_{\rm c}}{(2\pi)^4}\int \mathrm{d}^4p \frac{\Sigma(p)}{\Sigma^2(p) + p^2}\,, \tag{3.1}$$

which simplifies on making use of (2.19) and (2.20) to

$$\langle q\bar{q}\rangle_0 \tag{3.2}$$

$$= \frac{4N_{\rm c}}{(2\pi)^4}\int \mathrm{d}^4p \frac{\Sigma_0(1+x)}{(p^2 + m_0^2)[(1+x)^2 - 2y_0(1-y_0)]}\,,$$

where

$$x = p^2\beta\,; \quad y_0 = m_0^2\beta\,; \quad x_0 = \Sigma_0^2\beta\,. \tag{3.3}$$

The corresponding quantity f_π^2 may be defined in the chiral limit by

$$2f_\pi^2 P_\mu = \frac{N_{\rm c}}{(2\pi)^4}\int \mathrm{d}^4p\,\mathrm{Tr}[(\Sigma(p_1) + \Sigma(p_2))$$

$$\times\gamma_5 S_{\rm FR}(p_1)\mathrm{i}\gamma_\mu\gamma_5 S_{\rm FR}(-p_2)]\,,$$

where $S_{\rm FR}$ is given by (2.19) and (2.20), $P = p_1 + p_2$, and the pion–quark vertex function has been taken as [10]

$[\Sigma(p_1)+\Sigma(p_2)]/2f_\pi$. Fortunately the complete expression may be taken over from [10] (also given in [1]), viz.,

$$f_\pi^2 = \frac{4N_c}{(2\pi)^4}\int \mathrm{d}^4 p_E \frac{[1-(p^2/4)\partial_{p^2}]\Sigma^2(p)}{(\Sigma^2(p)+p^2)^2} \qquad (3.4)$$

in the Euclidean limit. The derivations of (3.2) and (3.4) are shown in Appendices A and B respectively. The final result for the condensate is summarized in (A.4–A.6) where the standard table of integrals [30] has often been employed. Similarly, for the pion decay constant, the final result is given by (B.5).

3.1 Results for condensate and pion decay

The key parameters of this theory are Σ_0, the constituent mass, and β, the parameter for the non-perturbative hadronic scale. A third quantity, the dynamical mass m_0, is determined by these via (2.17), which can be expressed in terms of the dimensionless parameters x_0 and y_0. Since the object of this investigation is not to provide a detailed phenomenological fit to these quantities, but rather to see if this new differential form of the DSE is consistent with the conventional range of values of the *constituent* mass, we shall refrain from any fine-tuning and offer some typical values within this alternative DSE framework, which is constrained by the fairly rigid connection between Σ_0 and m_0 brought about by the cubic equation (2.17). Thus, with a fixed Σ_0 at 300 MeV, Table 1 depicts some typical values of β, x_0 and y_0.

For these three sets we get under the MS scheme [28]

$$\langle q\bar{q}\rangle_0 = (0.1545; 0.0932; 0.114)\Sigma_0/\beta\,, \qquad (3.5)$$

where we have depicted the sensitivity of this quantity to the main parameters β for Σ_0 fixed at 300 MeV. For the values listed in Table 1, the numbers work out as

$$(359\,\mathrm{MeV})^3; (279\,\mathrm{MeV})^3; (284\,\mathrm{MeV})^3$$

respectively, suggesting that β should lie fairly close to its "critical" value $\beta_c = 1.646$, without further tuning. Similarly, the pionic constant works out for the three values of x_0 given above, as

$$f_\pi^2 = (92.0\,\mathrm{MeV})^2; (93.1\,\mathrm{MeV})^2; (94.3\,\mathrm{MeV})^2\,, \qquad (3.6)$$

respectively, with $\Sigma_0 = 300$ MeV. This quantity is not sensitive to β but varies as the square of Σ_0. These values give a rough test of this formalism without vastly extending the numerical framework. Note that the IR parameter λ at -0.064 has been rather passive in these determinations, but its temperature dependence is going to play a more active role in the T-dependent DSE, for the self-consistent determination of the critical temperature T_c to be considered in Sect. 4 to follow.

Table 1. Variations of x_0, y_0 with β

β	x_0	y_0
1.00	0.090	0.115
1.646	4/27	4/3
2.00	0.135	1.365

4 T-DSE in real time formalism

As noted in Sect. 1.1, since our DSE formulation departs from the more conventional thermodynamic formulations [15, 20] based on the free energy [15] or effective potential [20], we are not yet in a position to offer a complete set of critical indices near T_c, except the one for the T-dependence of the order parameter m_t. Keeping this in mind, to formulate the T-dependent DSE, we have two broad options: the real [22] versus imaginary [23] time formalisms. The $T=0$ structure of the DSE suggests that it is natural and convenient to employ the real time formalism and follow the prescription of Dolen–Jackiw [22] for adding to the quark and gluon propagators (which can be easily read off from the main DSE, (2.9)), the T-dependent imaginary parts of the Bose/Fermi types, leading to the modified propagators respectively as follows:

$$D_{FT}(k) = \frac{-\mathrm{i}}{k^2+m_g^2} + \frac{2\pi}{\exp\omega/T-1}\delta(k^2+m_g^2)\,;$$
$$\omega \equiv \sqrt{m_g^2+\mathbf{k}^2}\,, \qquad (4.1)$$

$$S_{FT}(p) = \frac{-\mathrm{i}}{\Sigma(p)+\mathrm{i}\gamma.p} - \frac{2\pi(\Sigma(p)-\mathrm{i}\gamma.p)}{\exp E_p/T+1}\delta\left(\Sigma^2(p)+p^2\right)\,, \qquad (4.2)$$

where the quark energy E_p is the fermionic analog of the gluon energy ω, (4.1). Taking the gluon case first, there are now two kinds of operations on (2.9). Namely, since the p^2 values are being considered to be on the mass shell, we shall now write $p^2 = -m_t^2$ (instead of $-m_0^2$) to emphasize the T-dependence of this quantity. Similarly (see Sect. 1) we shall consider the gluon mass m_g and the constituent mass Σ_0 to "run" with T, and designate them as Σ_t and m_{gt} respectively. Considering the bosonic and fermionic Boltzmann factors (4.1) and (4.2) in this order, we shall have extra contributions to the four pieces on the RHS of (2.9), but giving rise to 3D integrals only. We now collect these values separately, first upgrading the $T=0$ results of Sect. 2 to $T \neq 0$.

4.1 T-dependent $I; II$ and $I'; II'$

To simplify the four pieces of the DSE, (2.9), on the T-dependent mass shell, the following results are useful:

$$\Sigma(p) = m_t; \quad 2\Sigma'(p) - \Sigma''(p) = +\frac{\beta m_t^4}{\Sigma_t}\,; \qquad (4.3)$$

$$\Sigma(p) - \Sigma'(p)/2 = m_t - \frac{\beta m_t^4}{2\Sigma_t}\,. \qquad (4.4)$$

Collecting these results on the (now T-dependent) four pieces on the RHS of (2.9) we have

$$I + I' = \frac{+\beta m_t^4}{\Sigma_t} \times \left[2.9538 + \ln(2\zeta^2/m_t^2) + \frac{2\lambda_t}{m_t^2\beta}[\ln(m_t^2/2m_{gt}^2) - 1]\right],$$
$$II + II' = m_t - \frac{\beta m_t^4}{2\Sigma_t}\left(2 - \frac{8\beta m_t^3}{\Sigma_t}\right) \times \left[1 + \frac{2\lambda_t}{m_t^2\beta}[\ln(m_t^2/2m_{gt}^2) - 2]\right]. \tag{4.5}$$

To these pieces must be added the T-parts of the gluon propagators (bosonic) accruing from (4.1), and the T parts of the quark propagators (fermionic) from (4.2). These are basically 3D integrals because of the δ-functions. To evaluate them the following quantities come into play:

$$D(p-k) = \Sigma^2(p) + (p-k)^2 = -m_{gt}^2 + 2m_t\omega, \tag{4.6}$$
$$4\Sigma(p)\Sigma'(p) + 2p^2 - 2p.k = +\frac{4\beta m_t^5}{\Sigma_t} - 2m_t^2 + 2m_t\omega. \tag{4.7}$$

Here we have taken the rest frame of p_μ, viz., $\mathbf{p} = 0$. The bosonic T-parts normalized to the pieces in (4.5) are

$$\mathrm{BOSE}_T = 4\int \mathrm{d}\omega\sqrt{\omega^2 - m_{gt}^2}\frac{1}{\exp\omega/T - 1}[I_{\mathrm{BT}} + II_{\mathrm{BT}}] \tag{4.8}$$

where the lower limit of ω-integration is m_{gt}, and the two integrands are

$$I_{\mathrm{BT}} = -\frac{\beta m_t^4/\Sigma_t}{2\omega m_t - m_{gt}^2}; \tag{4.9}$$
$$II_{\mathrm{BT}} = \frac{m_t - \frac{\beta m_t^4}{2\Sigma_t}}{(2\omega m_t - m_{gt}^2)^2}[2m_t^2 - 2m_t\omega - 4m_t^5\beta/\Sigma_t]. \tag{4.10}$$

Similarly for the fermionic parts, denoted by $I_{(\mathrm{FT})}$ and $II_{(\mathrm{FT})}$, respectively. The complete T-dependent DSE is then obtained by modifying (2.9) à la (4.5) and adding the pieces (4.9) and (4.10), and the corresponding fermionic parts, after integrations. Before carrying out these integrations we notice some general features of these quantities in the neighborhood of the critical temperature T_c. Namely,
(i) the powers of m_t are spaced by *three* units;
(ii) m_t and m_{gt} are always involved in identical ratios.

One may infer from this that the critical index γ for both is the same at $3\gamma = 1$, consistent with universality [29] for such quantities. Thus, in the neighborhood of T_c one may take

$$[m_t; m_{gt}] \approx [\Sigma_0; m_g]\tau^\gamma; \quad \tau = 1 - T/T_\mathrm{c}; \quad \gamma = 1/3. \tag{4.11}$$

The T-dependence of m_0 may be handled via (2.17). Note that in the neighborhood of T_c, $\Sigma_t \approx m_t$, a result which is consistent with (7.278) of [29]. Retaining only the lowest powers of the small quantities m_t, m_{gt}, most of the terms in the T-DSE will drop out, and the integrals over (4.9) and (4.10) will lead to the net bosonic contribution

$$\mathrm{BOSE}_T/(4T) = \frac{m_t}{2m_{gt}(1 - m_{gt}^2/2m_t^2)} - [\ln(2T/m_{gt})]/2. \tag{4.12}$$

To this T-dependent (gluon propagator) contribution must be added the corresponding quark propagator contribution, (4.2), near $T = T_\mathrm{c}$, by following a procedure similar to the above one. For brevity, we indicate only the extra features, before writing the final result. The fermionic T-part of the quark propagator in (4.2) now becomes

$$(-2\mathrm{i}\pi)\frac{\delta((p-k)^2 + m_t^2)}{(\exp(E(\mathbf{p} - \mathbf{k}/T) + 1))}. \tag{4.13}$$

And analogous to (4.6),

$$k^2 + m_g^2 \approx 2m_t E_k - 2m_t^2 + m_{gt}^2; \quad E_k = \sqrt{(\mathbf{k}^2 + m_t^2)}. \tag{4.14}$$

Next, taking account of (4.3) and (4.4), and proceeding as in the gluon case, we can evaluate the quark counterpart of (4.8) in the neighborhood of $T = T_\mathrm{c}$ in the form

$$\mathrm{FERMI}_T \approx \left[-T\beta m_t^2 \tan^{-1}\left[\frac{T}{T + m_t}\right] + \lambda_t(-\gamma + \ln(T/m_t))/4\right] \Big/ (4\pi^2). \tag{4.15}$$

It is easily checked that this quark contribution is at least of $O(\sqrt{\beta}m_t)$ compared with the gluon one, so that it is justified to neglect it, at least near the critical point. The master equation T-DSE, keeping only the lowest order terms, now simplifies to

$$\frac{4\lambda_t}{\beta m_t}L_1 + \mathrm{BOSE}_T = 0; \quad L_1 = \ln(m_0^2/2m_g^2) - 2. \tag{4.16}$$

This equation suggests a simple structure for λ_t, perhaps one of the few that are consistent with its solution, viz.,

$$\lambda_t = \lambda_0(m_{gt}/m_g)^\gamma[-\ln\tau^\gamma + 1]; \quad \tau = 1 - T/T_\mathrm{c}, \tag{4.17}$$

where λ_0 may be identified with the value found in Sect. 3, viz., $\lambda = -0.064 \pm 0.003$, and the term unity in square brackets signifies its normalization at $T = 0$. Equation (4.16) after substitution from (4.12) now reduces to *two* equations, involving the coefficients of

$$\tau^\gamma; \quad \tau^\gamma \ln\tau^\gamma,$$

respectively, but we skip these equations for brevity. The result, after elimination of the quantity L_1 of (4.16) from them, and dividing out by T, is

$$-1/2 + (1/2\nu)[1 - \nu^2/2] = 1/2\ln[2T_\mathrm{c}/m_g]; \quad \nu = m_g/\Sigma_0, \tag{4.18}$$

using (4.11) near the critical point. Substituting from Sect. 2.3, viz., $\nu \approx 1/2$ gives the surprisingly simple result

$$\nu \approx 1/2 ; \quad 2T_{\mathrm{c}} \approx m_g \exp 7/8 , \tag{4.19}$$

leading to a reasonable value for the critical temperature, viz.,

$$T_{\mathrm{c}} \approx 180 \pm 20\,\mathrm{MeV} \quad (m_g = 150\,\mathrm{MeV}) . \tag{4.20}$$

4.2 Condensate and pionic constant near $T = T_{\mathrm{c}}$

For completeness we offer some brief comments on the predictions of this simple formalism on the corresponding T-dependent quantities $\langle q\bar{q}\rangle$ and f_π^2 near the critical point $T = T_{\mathrm{c}}$, analogously to the results of [1,5]. This is possible in view of the *analytical* expressions for these quantities as given in Appendices A and B in terms of y_0, a and x_0 respectively. In T-dependent form, $y_0 \sim m_t^2, a \sim m_t$, and $\Sigma_0 \sim m_t$, which in turn are expressible in terms of the basic "order" parameters m_t and m_{gt}; see (4.11). Substitution in (A.6) and (B.5) shows that $\langle q\bar{q}\rangle$ and f_π^2 tend to zero near the critical point like m_t and $m_t^2 \ln 1/m_t^2$ respectively, in general accord with standard expectations.

For comparison with other approaches, the chiral perturbation theory [6] for $f_\pi^(T)$ predicts [5]

$$f_\pi(T) = \tilde{f}_\pi \left[1 - \frac{N_f}{2\tilde{f}_\pi^2 (2\pi)^3} \int \mathrm{d}^3 p [\exp{(E/T)} - 1]^{-1}/E\right] ,$$

which however does not indicate how this quantity behaves near T_{c}. Another form [1] which is more in line with our parametrization of $\Sigma(p)$, suggests that Σ_t should vary as $\langle q\bar{q}\rangle_T$, in agreement with our result for the condensate.

5 Summary and conclusion

In retrospect, we have proposed a new (differential) form of the Dyson–Schwinger equation (DSE) for the mass function $\Sigma(p)$, based on an (infinitesimal) *subtractive* form of renormalization in QCD. Such a "subtraction" in turn amounts to employing a differential operator of the form $p_\mu \partial_\mu$ applied on both sides of the DSE, whose effect on the RHS is to *lower* the degree of divergence with respect to the integration variable k_μ by *two* units. It is in the background of this (differential form of) subtractive renormalization that it becomes possible to approximate the quantity $\Sigma(p-k)$ inside the integral by $\Sigma(p)$, since the k-dependence of this already decreasing quantity is no longer sensitive to the principal term $(p-k)^2$ in the quark propagator. [Without this background of an improved k-convergence, however, this approximation would not have been justified.] This crucial step, which has facilitated the integration over $\mathrm{d}^4 k$ without further ado, has thus helped to convert the DSE into a second-order differential equation, the extra order (beyond the rainbow approximation [9]) arising from the term responsible for satisfying the WT identity à la Pagels–Stokar [10], so as to preserve gauge invariance. To reinforce this effect, we have employed the Landau gauge which makes the DSE virtually dependent only on the mass function $\Sigma(p)$ by effectively eliminating the A-function [27]. (The "ghost terms" do not appear in this effective description.)

To solve the resulting differential form of the DSE, we have taken recourse to *three* crucial check-points: $p^2 = \infty$, $p^2 = 0$, and $p^2 = -m_0^2$, using a pole ansatz, (2.16) [1], which is consistent with the form p^{-2} in the large p^2 regime, in agreement with dynamical breaking of chiral symmetry for massless quarks [10], provided the argument of α_{s} is held fixed at some chosen value (here the UV parameter ζ). This has given a rather small value for the UV parameter ζ that appears as an argument of α_{s}, which effectively restricts the range of applicability of this formalism to moderate values of p^2 (perhaps adequate for the temperature range of interest for this paper). For the low p^2 regime, we have introduced two kinds of masses: the *constituent* mass $\Sigma(0)$ which is generally believed to be of $\sim 300\,\mathrm{MeV}$, and the *dynamical* mass m_0 which satisfies the equation $\Sigma(\mathrm{i}m_0) = m_0$ corresponding to the pole position of the quark propagator $p^2 = -m_0^2$ (see also [29]). Now for the simple form (2.16) the connection between the two "masses" is given by (2.18) (as the solution of a cubic equation) which corresponds to $\Sigma(0) < m_0$. The parameter β in (2.16), for a given $\Sigma(0)$, has been taken as a typical hadronic scale befitting the low energy regime of the DSE. The small IR parameter $2\mu^2$ ($\equiv \lambda/\beta$), which has played a passive role in the $T = 0$ description of the DSE, turns out to be rather crucial for $T > 0$, for which the ansatz (4.17) is necessary for a self-consistent solution of the T-DSE (see further below). We have also considered a non-zero value of the gluon mass for which several arguments have been advanced in the literature [13–15].

We have also carried out two important applications of $\Sigma(p)$ obtained from this new formulation of the DSE, viz., the quark condensate and the pion decay constant, more by way of some basic calibration of the formalism than as a means of detailed phenomenological fits to the hadronic data. Thus a fit to within $< 10\%$ has helped fix the parameters involved. After this check, we have attempted in Sect. 4 a T-dependent formulation of the DSE to see the extent to which it can simulate the critical temperature and at least one of the critical indices. To that end, we have taken the $p^2 = -m_t^2$ limit of the T-DSE near the critical point T_{c}, where it is small. In this respect, the demands of consistency have necessitated a T-dependence of the IR confining parameter λ, for which an ansatz of the form (4.17), calibrated to its value at $T = 0$, is indicated. Two clear results have emerged from the analysis, viz.,
(i) a bunching of the powers of m_t in units of *three* suggest a critical index $\gamma = 1/3$ according to the conventional analysis [29],
(ii) and the "matching" of the coefficients of like powers of the reduced temperature τ have led to a very simple solution of the form (4.18), leading to the reasonable T_{c} at $180 \pm 20\,\mathrm{MeV}$.

For a comparison of this result with those of contemporary approaches [15,20], our approach differs from these

in two important respects.
(i) Instead of starting from the free energy [15] or the effective potential [1] for appropriate differentiations to access the relevant thermodynamic quantities, we have used the T-DSE itself as the dynamical language to that end.
(ii) The role of an external H-field is sought to be partially simulated by the IR parameter λ which is necessarily T-dependent, instead of an approach by small but non-zero u–d masses [15, 20]. Further, in view of our explicit analytical expressions for $\langle q\bar{q}\rangle$ and f_π^2, we have also obtained analytic structures for their T-dependence, and we found indeed that they both vanish at the critical point, without a detailed numerical analysis [6, 15, 20]. However this approach has its weak points, especially the ad hoc nature of (4.17) for the T-dependence of the IR term. A second one is the lack of a more plausible understanding of the extent to which the IR term can substitute for the current masses [15, 20] to simulate the H-field effect. Attempts at throwing more light on these issues, as well as extending the T-DSE formalism to facilitate the evaluation of other critical indices [20, 29], are envisaged. Furthermore, in view of its central role, several other applications of the "mass function", such as $\pi \to 2\gamma$, and the e.m. pion form factor at finite temperature [31], are under way.

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Appendix A: evaluation of $\langle q\bar{q}\rangle_0$

Using the notation $x = \beta p^2$ and $y_0 = \beta m_0^2$, and anticipating a UV divergence, which requires a DR treatment [27], we write $4 \to n$ in (3.2), which reduces after the angular integration [11, 27] to

$$\langle q\bar{q}\rangle_0 = \frac{4N_c \pi^{n/2}\Sigma_0\zeta^\epsilon}{(2\pi)^n \Gamma(n/2)\beta^{n/2-1}} \int_0^\infty \mathrm{d}x x^{n/2-1} F(x)\,, \quad \text{(A.1)}$$

where

$$F(x) = \frac{(1+x)}{(x+y_0)[(1+x)^2 - a^2]}\,; \quad a^2 = 2y_0(1-y_0)\,. \quad \text{(A.2)}$$

Now break up $F(x)$ into partial fractions

$$F(x) = \frac{1}{[(1-y_0)^2 - a^2]} \times \left[\frac{1-y_0}{x+y_0} - \frac{1-y_0+a}{2(1+x-a)} - \frac{1-y_0-a}{2(1+x+a)}\right].$$

The integration of each term above is carried out according to

$$\int_0^\infty x^{n/2-1}\mathrm{d}x/(A+x) = A^{1-\epsilon/2}\Gamma(n/2)\Gamma(1-n/2)\,. \quad \text{(A.3)}$$

The rest is a matter of collecting all three terms after giving a DR [27] treatment to each. The final result is

$$\begin{aligned}\langle q\bar{q}\rangle_0 &= \frac{\Sigma_0 N_c}{\beta 4\pi^2} \times [g(a)(\ln 4\pi - \gamma + 1 + \ln\zeta^2\beta) + h(a)]\,;\\ g(a) &= \frac{1-y_0-a^2/2}{(1-y_0)^2 - a^2}\,; \qquad \text{(A.4)}\\ h(a) &= \frac{y_0(1-y_0)\ln y_0}{(1-y_0)^2-a^2} - (1/2)\sum_\pm \frac{(1\pm a)\ln(1\pm a)}{1-y_0\pm a}\,.\end{aligned}$$

This result is valid for small y_0 (i.e., $\beta \sim 1$), when $a^2 > 0$. However for larger β, vide (2.18) and (2.19), y_0 exceeds unity, and $a^2 < 0$. For such cases, put $a^2 = -b^2$. In particular the partial fraction break-up for β_c, corresponding to $y_0 = 4/3$, is rather simple:

$$F(x) = \frac{b^2 + (1+x)/3}{(1+x)^2 + b^2} - \frac{1/3}{x+y_0}\,,$$

since $b^2 + (y_0-1)^2$ becomes unity. Now using the result [30]

$$\int_0^\infty \frac{x^{n/2-1}\mathrm{d}x}{1+b^2+2x+x^2} = -\frac{(1+b^2)^{n/4-1}\sin(n/2-1)t}{\sin t \sin n\pi/2}\,, \quad \text{(A.5)}$$

where

$$\cos t = +1/\sqrt{1+b^2}\,; \quad \sin t = +b/\sqrt{1+b^2}\,,$$

and giving a DR treatment [27] as above, the result corresponding to (A.5) is

$$\begin{aligned}\langle q\bar{q}\rangle_0 &= \frac{\Sigma_0 N_c}{\beta 4\pi^2}[b^2(\ln 4\pi - \gamma + 1 + \ln\zeta^2\beta) + f(b)]\,;\\ f(b) &= (1/6 - b^2/2)\ln(1+b^2) - \ln y_0/3 - (4b/3)\tan^{-1} b\,. \qquad \text{(A.6)}\end{aligned}$$

For purposes of obtaining the temperaure dependence of the quark condensate (to be discussed in Sect. 4), we record the results of these integrations in the limit of small a and y_0, for which (A.4) is appropriate:

$$g(a) \approx 1 + O(y_0)\,; \quad h(a) \approx y_0 \ln y_0 + a^2 \sim a\ln a\,. \quad \text{(A.7)}$$

Substitution in (A.1) gives in this limit

$$\langle q\bar{q}\rangle_0 = \frac{\Sigma_0 N_c}{\beta 4\pi^2}[(\ln 4\pi - \gamma + 1 + \ln\zeta^2\beta) + O(a\ln a)]\,, \quad \text{(A.8)}$$

which lends itself immediately to a finite T treatment in the neighborhood of the critical point (see Sect. 4).

Appendix B: evaluation of f_π^2

Since the integral (3.4) is convergent by itself, DR [27] is not needed in this case. After the angular integrations (using the dimensionless units x, y_0 as before) and carrying out the differentiations, (3.4) reduces to

$$f_\pi^2 = \frac{\Sigma_0^2 N_c}{4\pi^2}\mathcal{I} \tag{B.1}$$

where the integral is defined by

$$\mathcal{I} = \int_0^\infty \mathrm{d}x x \frac{(1+x)(1+3x/2)}{[x_0 + x(1+x)^2]^2}\,. \tag{B.2}$$

Now transform the variable from x to u,

$$u = \frac{x}{1+x}\,; \quad 0 \le u \le 1\,.$$

The result of this is to give an integral in u:

$$\mathcal{I} = \int_0^1 \frac{\mathrm{d}uu(1-u)(1+u/2)}{[x_0(1-u)^3 + u]^2}\,. \tag{B.3}$$

While this integral can in principle exactly be performed, it is instructive to obtain an approximate *analytical* expression which in practice is sufficiently accurate, so as to lend itself to a generalization to *finite* temperatures (see below). The trick lies in the observation that most of the contributions arise from the region of *small* values of u. Then (B.3) simplifies to

$$\mathcal{I} \approx \int_0^1 \frac{\mathrm{d}uu(1-u/2)}{[x_0(1-3u)+u]^2}\,.$$

Now integration by parts gives the final result,

$$\mathcal{I} = \frac{1/2}{(1-3x_0)^2}\ln(1-2x_0)/x_0 - \frac{1/2}{(1-2x_0)(1-3x_0)}\,. \tag{B.4}$$

Unlike the case of $\langle q\bar{q}\rangle$, this result is valid for all allowed x_0. For purposes of determining the temperature dependence of f_π^2, to be discussed in Sect. 4, we record, as in Appendix A, the corresponding results for small x_0. This gives

$$\mathcal{I} \approx 1/2 \ln 1/x_0 - 1/2\,.$$

Substitution in (B.1) leads to

$$f_\pi^2 \approx \frac{\Sigma_0^2 N_c}{8\pi^2(1-3x_0)^2}\left[\ln 1/x_0 - \frac{1-3x_0}{1-2x_0}\right]\,, \tag{B.5}$$

which lends itself immediately to a finite T treatment in the vicinity of the critical point T_c (see Sect. 4).

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